

Lecture 11: Graphs of Functions

Definition If f is a function with domain A , then the **graph** of f is the set of all ordered pairs

$$\{(x, f(x)) | x \in A\},$$

that is, the graph of f is the set of all points (x, y) such that $y = f(x)$. This is the same as the graph of the equation $y = f(x)$, discussed in the lecture on Cartesian co-ordinates.

The graph of a function allows us to translate between algebra and pictures or geometry.

A function of the form $f(x) = mx + b$ is called a **linear function** because the graph of the corresponding equation $y = mx + b$ is a line. A function of the form $f(x) = c$ where c is a real number (a constant) is called a **constant function** since its value does not vary as x varies.

Example Draw the graphs of the functions:

$$f(x) = 2, \quad g(x) = 2x + 1.$$

Graphing functions As you progress through calculus, your ability to picture the graph of a function will increase using sophisticated tools such as limits and derivatives. The most basic method of getting a picture of the graph of a function is to use the join-the-dots method. Basically, you pick a few values of x and calculate the corresponding values of y or $f(x)$, plot the resulting points $\{(x, f(x))\}$ and join the dots.

Example Fill in the tables shown below for the functions

$$f(x) = x^2, \quad g(x) = x^3, \quad h(x) = \sqrt{x}$$

and plot the corresponding points on the Cartesian plane. Join the dots to get a picture of the graph of each function.

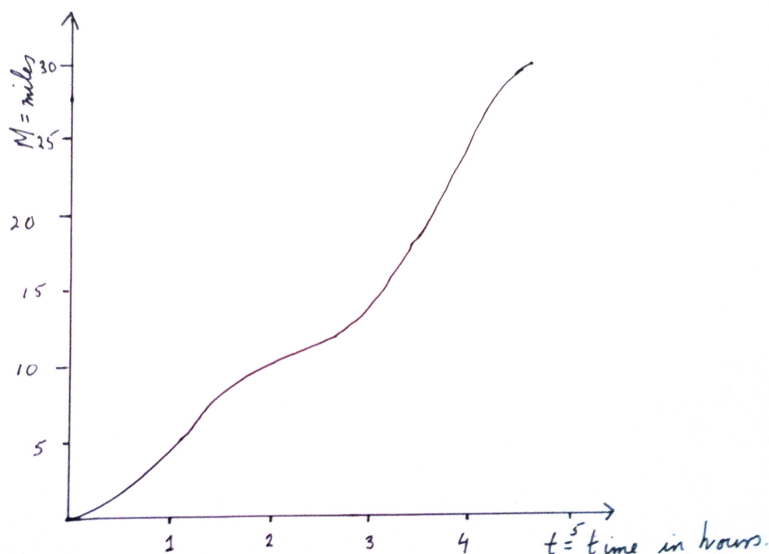
x	$f(x) = x^2$	x	$g(x) = x^3$	x	$h(x) = \sqrt{x}$
-3		-3		0	
-2		-2		1	
-1		-1		4	
0		0		9	
1		1		16	
2		2		25	
3		3		36	

Graph of $f(x) = 1/x$

x	$f(x) = 1/x$	x	$f(x) = 1/x$
-100		-1/10	
-10		-1/100	
-1		-1/1000	
0?		0?	
1		1/1000	
10		1/100	
100		1/10	

Getting Information From the Graph of a function

Suppose the following graph shows the distance a runner in a 30 mile race has covered t hours after the beginning of a race.

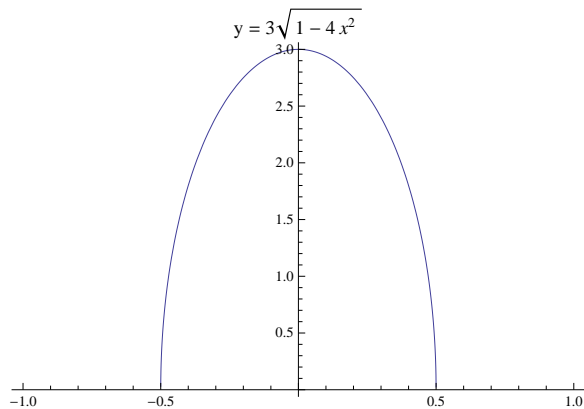


- (a) Approximately how much distance has the runner covered after 1 hour?
- (b) Approximately how long does it take for the runner to complete the course (30 miles)?

Domain and Range on Graph

The **domain** of the function f is the set of all values of x for which f is defined and this corresponds to all of the x -values on the graph in the xy -plane. The **range** of the function f is the set of all values $f(x)$ which corresponds to the y values on the graph in the xy -plane.

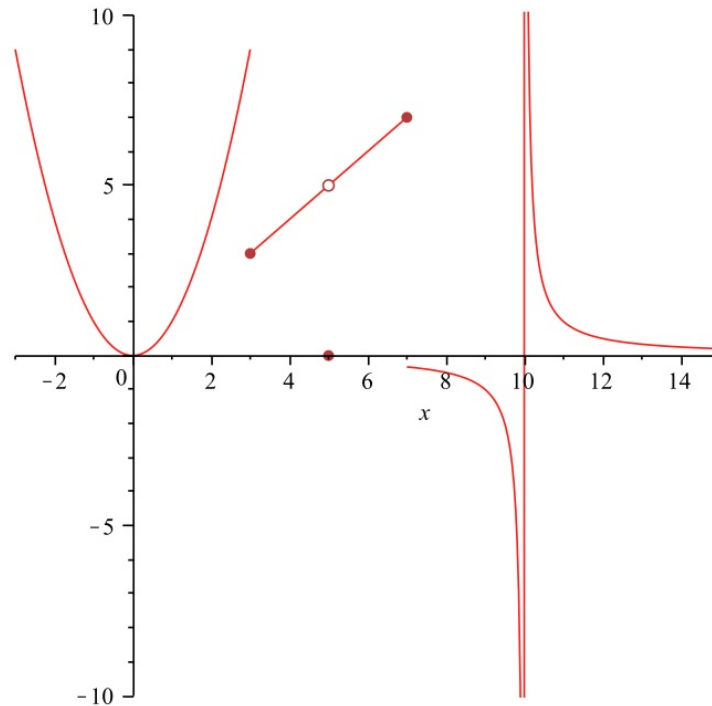
Example Use the graph shown below to find the domain and range of the function $f(x) = 3\sqrt{1 - 4x^2}$.



Graphing Piecewise defined functions

Recall that a piecewise defined function is typically defined by different formulas on different parts of its domain. The graph, therefore consists of separate pieces as in the example shown below.

$$k(x) = \begin{cases} x^2 & -3 < x < 3 \\ x & 3 \leq x < 5 \\ 0 & x = 5 \\ x & 5 < x \leq 7 \\ \frac{1}{x-10} & x > 7 \end{cases}$$



- We use a solid point at the end of a piece to emphasize that that point is on the graph. For example, the point $(3, 3)$ is on the graph here, whereas the point above it, $(3, 9)$, at the end of the portion of the graph of $y = x^2$ is not.
- We use a circle to denote that a point is excluded. For example the value of this function at 5 is 0, therefore the point $(5, 0)$ is on the graph as indicated with the solid dot. The point above it on the line $y = x$, $(5, 5)$, is not on the graph and is excluded from the graph. We indicate this with a circle at the point $(5, 5)$.
- Note that the formula $y = \frac{1}{x-10}$ does not make sense when $x = 10$. Therefore $x = 10$ is not in the domain of this function. As the values of x get closer and closer to 10 from above, the values of $\frac{1}{x-10}$ get larger and larger. Therefore the y values on the graph approach $+\infty$ as we approach $x = 10$ from the right. On the other hand the y values on the graph approach $-\infty$ as x approaches 10 from the left. Although there is no point on the graph at $x = 10$, the (computer generated) graph shows a vertical line at $x = 10$. This line is called a **vertical asymptote** to the graph and we will discuss such asymptotes in more detail in calculus.

Example Graph the piecewise defined function

$$f(x) = \begin{cases} x & -\infty < x \leq 1 \\ 2x & 1 < x < 2 \\ 1 & x = 2 \\ x^2 & x > 2 \end{cases}$$

Example Graph the absolute value function

$$g(x) = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$$

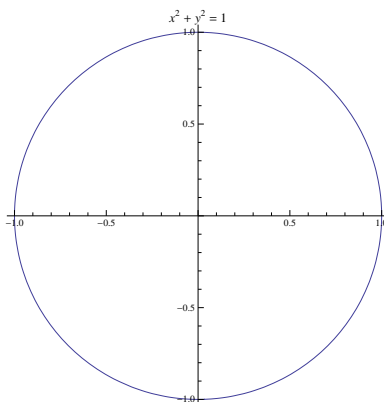
Graphs of Equations; Vertical Line Test

it is important in calculus to distinguish between the graph of a function and graphs of equations which are not the graphs of functions. We will develop a technique called implicit differentiation to allow us to compute derivatives at (some) points on the graphs of equations which are not graphs of functions. It is therefore important to be fully aware of the relationship between graphs of equations and graphs of functions.

Recall that the defining characteristic of a function is that for every point in the domain, we get exactly one corresponding point in the range. This translates to a geometric property of the graph of the function $y = f(x)$, namely that for each x value on the graph we have a unique corresponding y value. This in turn is equivalent to the statement that if a vertical line of the form $x = a$ cuts the graph of $y = f(x)$, it cuts it exactly once. Therefore we get a geometric property which characterizes the graphs of functions:

Vertical Line Property A curve in the xy -plane is the graph of a function if and only if no vertical line intersects the curve more than once.

Recall that the graph of an equation in x and y is the set of all points (x, y) in the plane which satisfy the equation. For example the graph of the equation $x^2 + y^2 = 1$ is the unit circle (circle of radius 1 centered at the origin).

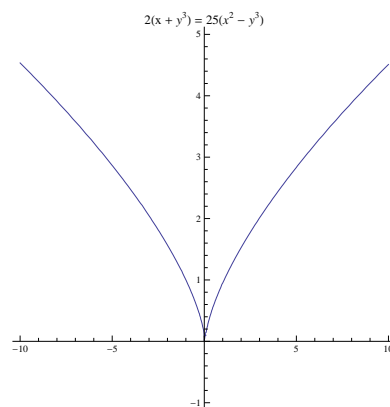
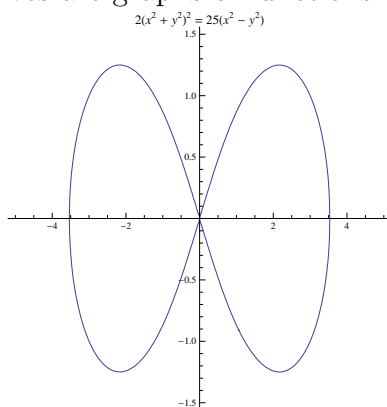
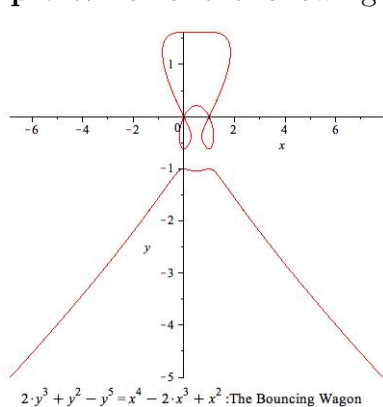


If we can solve for y uniquely in terms of x in the given equation, we can rearrange the equation to look like $y = f(x)$ for some function of x . Rearranging the equation does not change the set of points which satisfy the equation, that is, it does not alter the graph of the equation. So **being able to solve for y uniquely in terms of x is the algebraic equivalent of the graph of the equation being the graph of a function.** This is equivalent to the graph of the equation having the vertical line property given above.

Vertical Line Test The graph of an equation is the graph of a function (or equivalently if we can solve for y uniquely in terms of x) if no vertical line cuts the curve more than once.

More generally, this applies to graphs given in pieces which may be the graph of a piecewise defined function. One or several curves in the xy plane form the graph of a function (possibly piecewise defined) if no vertical line cuts the collection of curves more than once.

Example Which of the following curves are graphs of functions?



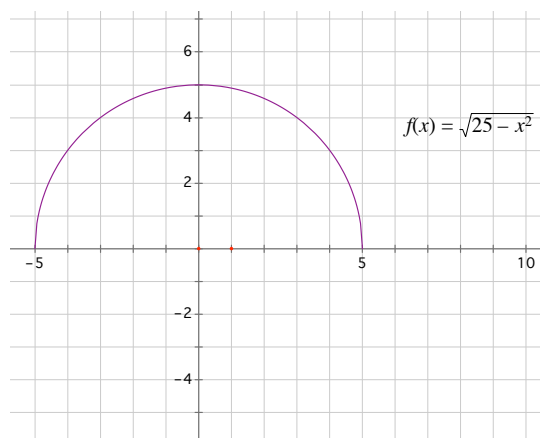
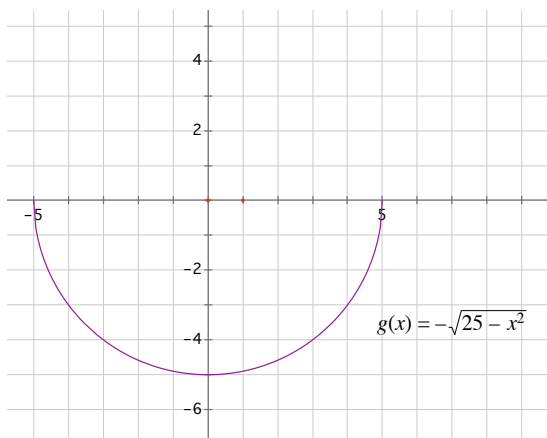
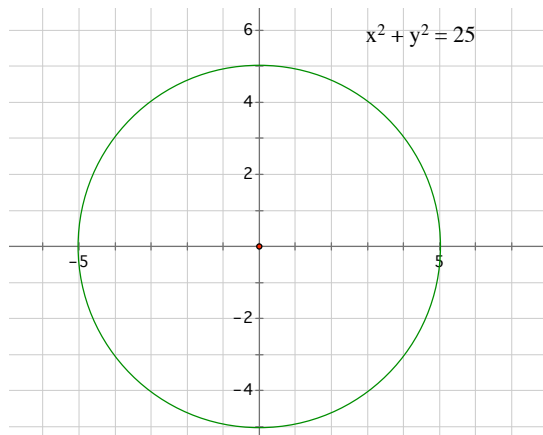
Let us see what happens if we try to solve for y in an equation which describes a curve which does not pass the vertical line test. If we try to solve for y in terms of x in the equation

$$x^2 + y^2 = 25$$

we get 2 new equations

$$y = \sqrt{25 - x^2} \quad \text{and} \quad y = -\sqrt{25 - x^2}.$$

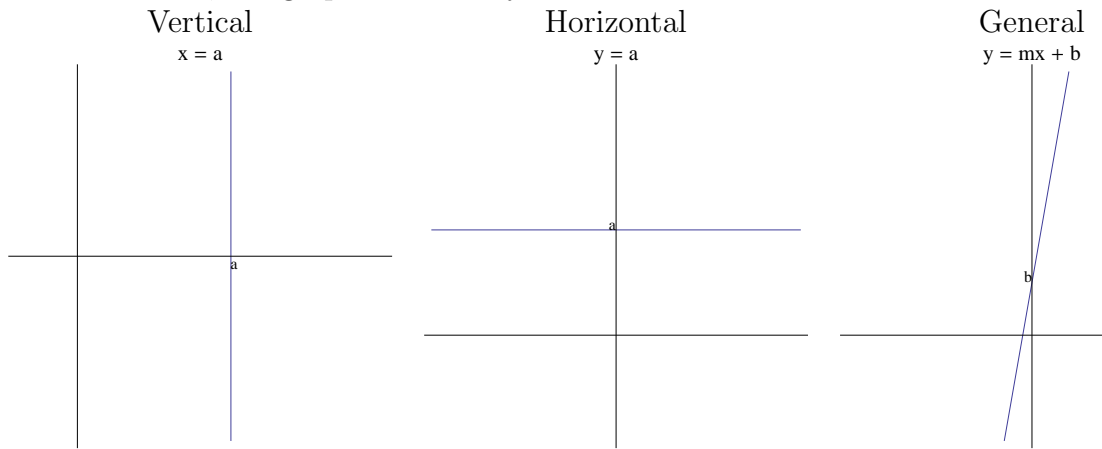
The graph of the equation $x^2 + y^2 = 25$ is a circle centered at the origin $(0, 0)$ with radius 5 and the above two equations describe the upper and lower halves of the circle respectively.



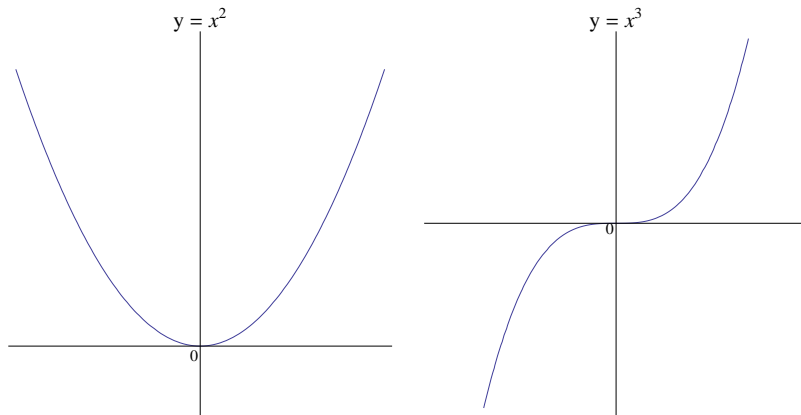
The graphs of the upper and lower halves of the circle are the graphs of functions, but the circle itself is not.

Here is a catalogue of basic functions, the graphs of which you should memorize for future reference:

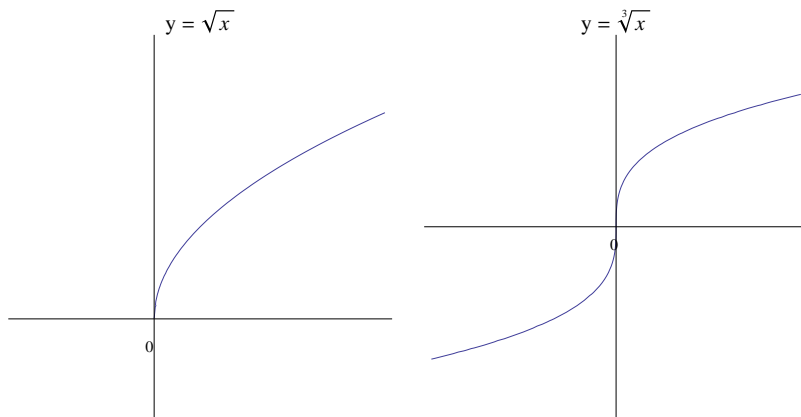
Lines



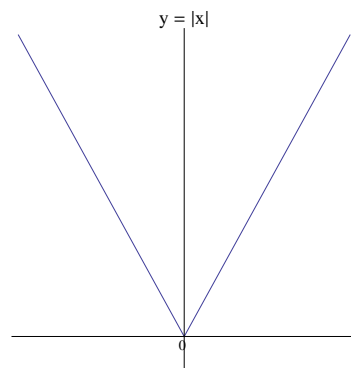
Power Functions



Root Functions



Absolute Value Function



$y = 1/x$

